An $\mathcal{H}_2 \otimes \mathcal{L}_2$-Optimal Model Order Reduction Approach for Parametric Linear Time-Invariant Systems

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So far, $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal model order reduction (MOR) of linear time-invariant systems, preserving the affine parameter dependence, was only considered for special cases by Baur et al in 2011. In this contribution, we present necessary conditions for an $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal parametric reduced order model, for general affine parametric systems resembling the special case investigated by Baur et al.

1 Introduction

Various mathematical and physical processes can be modeled as parametric linear time-invariant (LTI) systems

$$E(\mu)\dot{x}(t) = A(\mu)x(t) + B(\mu)u(t), \quad y(t) = C(\mu)x(t), \qquad (\Sigma(\mu; E; A, B, C))$$

including e.g. geometrical or physical parameters $\mu \in M \subset \mathbb{R}^d$, where $E(\mu), A(\mu) \in \mathbb{R}^{n \times n}$ are the system matrices, $B(\mu) \in \mathbb{R}^{n \times p}$ and $C(\mu) \in \mathbb{R}^{q \times n}$ are input and output matrices, respectively. Furthermore, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the input and $y(t) \in \mathbb{R}^q$ is the output.

Our aim is to find a reduced order model (ROM)

$$\widehat{E}(\mu)\dot{\widehat{x}}(t) = \widehat{A}(\mu)\widehat{x}(t) + \widehat{B}(\mu)u(t), \quad \widehat{y}(t) = \widehat{C}(\mu)\widehat{x}(t), \qquad (\widehat{\Sigma}(\mu; \widehat{E}; \widehat{A}, \widehat{B}, \widehat{C}))$$

of order $r \ll n$ preserving structural properties and approximating the dynamical behavior of $(\Sigma(\mu; E; A, B, C))$ by minimizing the $\mathcal{H}_2 \otimes \mathcal{L}_2$-error of the transfer functions

$$H(s, \mu) = C(\mu)\left(sE(\mu) - A(\mu)\right)^{-1} B(\mu), \qquad \widehat{H}(s, \mu) = \widehat{C}(\mu)\left(s\widehat{E}(\mu) - \widehat{A}(\mu)\right)^{-1} \widehat{B}(\mu),$$

with $s \in \mathbb{C}$ and $\mu \in M$, describing the relation between input and output in the frequency domain. The error system $\Sigma_{\text{err}}(\mu; E_{\text{err}}; A_{\text{err}}, B_{\text{err}}, C_{\text{err}})$ given by the matrices $E_{\text{err}}(\mu) = \begin{bmatrix} E(\mu) & \\ & \widehat{E}(\mu) \end{bmatrix}$, $A_{\text{err}}(\mu) = \begin{bmatrix} A(\mu) & \\ & \widehat{A}(\mu) \end{bmatrix}$, $B_{\text{err}}(\mu) = \begin{bmatrix} B(\mu) \\ \widehat{B}(\mu) \end{bmatrix}$ and $C_{\text{err}}(\mu) = \begin{bmatrix} C(\mu) & -\widehat{C}(\mu) \end{bmatrix}$ then has the transfer function $H_{\text{err}}(s, \mu) = H(s, \mu) - \widehat{H}(s, \mu)$ and the $\mathcal{H}_2 \otimes \mathcal{L}_2$-error is given as (see [1] and [2])

$$\|H_{\text{err}}\|^2_{\mathcal{H}_2 \otimes \mathcal{L}_2(M)} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_M \|H_{\text{err}}(\mathrm{i}\omega, \mu)\|_F^2 \,\mathrm{d}\mu \,\mathrm{d}\omega = \int_M \operatorname{tr}\left(C_{\text{err}}(\mu) P_{\text{err}}(\mu) C_{\text{err}}(\mu)^T\right) \mathrm{d}\mu, \qquad (1)$$

where $P_{\text{err}}(\mu)$ is the controllability Gramian of the error system and given as the solution of

$$A_{\text{err}}(\mu) P_{\text{err}}(\mu) E_{\text{err}}(\mu)^T + E_{\text{err}}(\mu) P_{\text{err}}(\mu) A_{\text{err}}(\mu)^T = -B_{\text{err}}(\mu) B_{\text{err}}(\mu)^T. \qquad (2)$$

By the usual duality argument the error can also be computed using the observability Gramian $Q_{\text{err}}(\mu)$ of the error system.

2 $\mathcal{H}_2 \otimes \mathcal{L}_2$-Optimality Conditions

Here, we assume all matrices of systems $(\Sigma(\mu; E; A, B, C))$ and $(\widehat{\Sigma}(\mu; \widehat{E}; \widehat{A}, \widehat{B}, \widehat{C}))$ to be affine decomposable, i.e.

$$E(\mu) = \sum_{i=1}^{m_E} e_i(\mu) E_i, \quad A(\mu) = \sum_{j=1}^{m_A} a_j(\mu) A_j, \quad B(\mu) = \sum_{k=1}^{m_B} b_k(\mu) B_k, \quad C(\mu) = \sum_{l=1}^{m_C} c_l(\mu) C_l,$$

$$\widehat{E}(\mu) = \sum_{i=1}^{m_E} e_i(\mu) \widehat{E}_i, \quad \widehat{A}(\mu) = \sum_{j=1}^{m_A} a_j(\mu) \widehat{A}_j, \quad \widehat{B}(\mu) = \sum_{k=1}^{m_B} b_k(\mu) \widehat{B}_k, \quad \widehat{C}(\mu) = \sum_{l=1}^{m_C} c_l(\mu) \widehat{C}_l,$$

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with continuous functions $e_i, a_j, b_k, c_l \colon M \to \mathbb{R}$ and matrices $E_i, A_j \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times p}$, $C_l \in \mathbb{R}^{q\times n}$, $\widehat{E}_i, \widehat{A}_j \in \mathbb{R}^{r\times r}$, $\widehat{B}_k \in \mathbb{R}^{r\times p}$ and $\widehat{C}_l \in \mathbb{R}^{q\times r}$ for $i \in [m_E] := \{1, 2, \dots, m_E\}$, $j \in [m_A]$, $k \in [m_B]$, $l \in [m_C]$. The parameter domain $M \subset \mathbb{R}^d$ is assumed to be compact. Furthermore, we assume $E(\mu)$ to be uniformly invertible, i.e. invertible for all parameters $\mu \in M$, and the system $(\Sigma(\mu; E; A, B, C))$ to be uniformly asymptotically stable, i.e. for all $\mu \in M$ the matrix pencil $\lambda E(\mu) - A(\mu)$ has all eigenvalues in the open left half of the complex plane.

2.1 General case

Minimizing (1) leads to an optimization problem with constraint (2). This can be solved applying the Lagrangian multiplier method ending up with $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal necessary conditions:

Theorem 2.1 *Let $\widehat{\Sigma}(\mu; \widehat{E}; \widehat{A}, \widehat{B}, \widehat{C})$ be an $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal ROM for $\Sigma(\mu; E; A, B, C)$. Further, let the Gramians of the error system $\Sigma_{err}(\mu; E_{err}; A_{err}, B_{err}, C_{err})$ be given as $P_{err}(\mu) = \begin{bmatrix} P(\mu) & \widetilde{P}(\mu) \\ \widetilde{P}(\mu)^T & \widehat{P}(\mu) \end{bmatrix}$ and $Q_{err}(\mu) = \begin{bmatrix} Q(\mu) & \widetilde{Q}(\mu) \\ \widetilde{Q}(\mu)^T & \widehat{Q}(\mu) \end{bmatrix}$, with $P(\mu), Q(\mu), \widehat{P}(\mu), \widehat{Q}(\mu)$ the corresponding Gramians of $\Sigma(\mu; E; A, B, C)$ and $\widehat{\Sigma}(\mu; \widehat{E}; \widehat{A}, \widehat{B}, \widehat{C})$. Then, it holds*

$$
\begin{aligned}
0 &= \int_M e_i(\mu) \left(\widetilde{Q}(\mu)^T A(\mu) \widetilde{P}(\mu) + \widehat{Q}(\mu)^T \widehat{A}(\mu) \widehat{P}(\mu) \right) \mathrm{d}\mu, \quad i \in [m_E], \\
0 &= \int_M a_j(\mu) \left(\widetilde{Q}(\mu)^T E(\mu) \widetilde{P}(\mu) + \widehat{Q}(\mu)^T \widehat{E}(\mu) \widehat{P}(\mu) \right) \mathrm{d}\mu, \quad j \in [m_A], \\
0 &= \int_M b_k(\mu) \left(\widetilde{Q}(\mu)^T B(\mu) + \widehat{Q}(\mu)^T \widehat{B}(\mu) \right) \mathrm{d}\mu, \quad k \in [m_B], \\
0 &= \int_M c_l(\mu) \left(C(\mu) \widetilde{P}(\mu) - \widehat{C}(\mu) \widehat{P}(\mu) \right) \mathrm{d}\mu, \quad l \in [m_C].
\end{aligned} \tag{3}
$$

A Two-Sided Iteration Algorithm (TSIA, see [3]) type MOR method for general systems $\Sigma(\mu; E; A, B, C)$ solving (3) is subject to ongoing research.

2.2 Resemblance to [1]

In [1], system matrices E and A are assumed to be constant, while input and output matrices have one parameter each, living in a parameter domain $M = [0, 1]^2$. In this paper, it was shown, that the $\mathcal{H}_2 \otimes \mathcal{L}_2$-error is equivalent to the $\mathcal{H}_2$-error of a weighted non-parametric multiple-input multiple-output (MIMO) system. Thus, a $\mathcal{H}_2 \otimes \mathcal{L}_2$-optimal ROM was computed by solving the non-parametric $\mathcal{H}_2$-optimality problem using an interpolatory method, the Iterative Rational Krylov Algorithm (IRKA).

Using the special model structure in [1] in our work, $P_{\text{err}}(\mu)$ and hence the dual Gramian $Q_{\text{err}}(\mu)$ can be written as affine decompositions. This representation allows us to generate a ROM without computing the integral matrix equations (3) explicitly. Picking up the idea of TSIA for non-parametric LTI systems, an improved ROM can be computed iteratively from a given ROM by solving non-parametric matrix equations and projecting with the orthogonalized solutions as in non-parametric MOR methods. These solutions turn out to be the Gramians of the above mentioned weighted MIMO system.

Thus, the optimization problem solved in [1] by reformulating the problem and using IRKA can also be solved using a parametric variant of TSIA.

3 Conclusion

In this paper, we present $\mathcal{H}_2 \otimes \mathcal{L}_2$-necessary optimality conditions for parametric LTI systems. These generalize results from [1] for the case of non-parametric E and A and, in that case, gives rise to a parametric TSIA as an alternative to IRKA. An efficient method based on conditions (3) for general affine parameter dependencies, however, is still an open problem.

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References

[1] U. Baur, C. A. Beattie, P. Benner, and S. Gugercin, SIAM J. Sci. Comput. **33**(5), 2489–2518 (2011).
[2] A. C. Antoulas, Approximation of Large-Scale Dynamical Systems, Adv. Des. Control, Vol. 6 (SIAM Publications, Philadelphia, PA, 2005).
[3] Y. Xu and T. Zeng, Int. J. Numer. Anal. Model. **8**(1), 174–188 (2011).